Scattering From a Sphere of Small Radius Embedded Into a Dielectric One

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Abstract—**In this paper, the scattering of a plane electromagnetic wave from a metallic or dielectric sphere of electrically small radius, embedded into a dielectric one, is considered. The classical method of separation of variables is used, combined with translational addition theorems for spherical vector wave functions. Analytical expressions are obtained for the scattered field and the various scattering cross-sections, in the case of an inner sphere with electrically small radius. Numerical results are given for various values of the parameters and for metallic and dielectric inner sphere. Some remarks are made about the possibility of detection or identification of inhomogeneities or nonsymmetries.**

I. Introduction

SCATTERING from composite bodies can give information for their internal composition. Thus, by observing their scattered field one can detect inner inhomogeneities, nonsymmetries, etc. The shape of the boundaries of such bodies severely limits the possibility for analytical solution of the scattering problem. Various numerical techniques are usually used for complicated geometries, a few examples of which appear in the recent papers [1]–[6]. Also, perturbational methods can be used, like those appearing in [7]–[9], for example.

In the present paper, the scattering of a plane electromagnetic wave by a metallic or dielectric sphere of electrically small (much smaller than the wavelength) radius, embedded into a dielectric one, is considered (see Fig. 1). A special analytical perturbation method is used in order to obtain approximate (first order) analytical expressions for the scattered field and the various scattering cross-sections. This method was initially developed for the scalar problem of scattering from an infinite cylinder of small radius embedded into a dielectric one [10]. The present vector problem appears much more complex and very lengthy in algebraic manipulation than the previous scalar one. The method of separation of variables is used, together with translational addition theorems for spherical vector wave functions.

The scattering of a plane electromagnetic wave by two arbitrary spheres outside of each other was solved elsewhere [11], [12].

The case of a metallic inner sphere with small (electrically) radius is examined in Section II, while that of a small dielectric inner sphere is examined in Section III. All metallic or dielectric materials are lossless. Finally, in Section IV, numerical results for various scattering cross-sections are given for various values of the parameters. Also, some remarks are made about the possibility of detection or identification of inhomogeneities or nonsymmetries.

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II. Metallic Sphere of Small Radius Embedded into a Dielectric One

In this section, we examine the scattering of a plane electromagnetic wave from a perfectly conducting metallic sphere of small radius, coated eccentrically by a dielectric one. The geometry of the problem is shown in Fig. 1. The outer and inner sphere radii are R_1 and R_2, respectively, while d is the distance between their centers O_1 and O_2, which are origins of two cartesian coordinate systems with parallel axes. The origin O_2 lies at the general position (d, θ_O, φ_O) of the system $O_1x_1y_1z_1$. The dielectric constant, the magnetic permeability, and the wavenumber are ϵ_1, μ_1, k_1 and ϵ_2, μ_2, and k_2 in regions 1 and 2, respectively. The materials of both regions are lossless. Region 3 in the present case is perfectly conducting.

Let $\vec{E}^{inc}$ be the electric field intensity of an incident plane electromagnetic wave travelling in the $+z$ direction that impinges on the scatterer of Fig. 1. This intensity has the expression [13], [14]:

$$\begin{aligned}\vec{E}^{inc} &= \vec{E}_o e^{ik_1z_1}\\ &= -(E_{ox}+iE_{oy})\sum_{n=1}^{\infty} i^{n+1}\frac{2n+1}{2}\\ &\quad\cdot[\vec{m}^{(1)}_{-1n}(k_1r_1,\theta_1,\varphi_1)-\vec{n}^{(1)}_{-1n}(k_1r_1,\theta_1,\varphi_1)]\\ &\quad-(E_{ox}-iE_{oy})\sum_{n=1}^{\infty} i^{n+1}\frac{2n+1}{2n(n+1)}\\ &\quad\cdot[\vec{m}^{(1)}_{1n}(k_1r_1,\theta_1,\varphi_1)\\ &\quad+\vec{n}^{(1)}_{1n}(k_1r_1,\theta_1,\varphi_1)]\end{aligned}\tag{1}$$

where $\vec{E}_o = E_{ox}\hat{x}+E_{oy}\hat{y}(|\vec{E}_o|=1)$ defines the direction of polarization of the wave and $\vec{m}^{(1)}_{mn}$, $\vec{n}^{(1)}_{mn}$, $(m=\pm1)$ are the complex spherical eigenvectors that can be found in [14], [15]. These eigenvectors are expressed with respect to the origin O_1. The magnetic field intensity of the incident wave is

$$\begin{aligned}\vec{H}^{inc} &= -\frac{i}{\zeta_1}(E_{ox}+iE_{oy})\sum_{n=1}^{\infty} i^{n+1}\frac{2n+1}{2}\\ &\quad\cdot[\vec{m}^{(1)}_{-1n}(k_1r_1,\theta_1,\varphi_1)-\vec{n}^{(1)}_{-1n}(k_1r_1,\theta_1,\varphi_1)]\\ &\quad+\frac{i}{\zeta_1}(E_{ox}-iE_{oy})\sum_{n=1}^{\infty} i^{n+1}\frac{2n+1}{2n(n+1)}\\ &\quad\cdot[\vec{m}^{(1)}_{1n}(k_1r_1,\theta_1,\varphi_1)+\vec{n}^{(1)}_{1n}(k_1r_1,\theta_1,\varphi_1)],\\ &\qquad\zeta_1=(\mu_1/\epsilon_1)^{1/2}\end{aligned}\tag{2}$$

The time factor $\exp(-i\omega t)$ is omitted everywhere.

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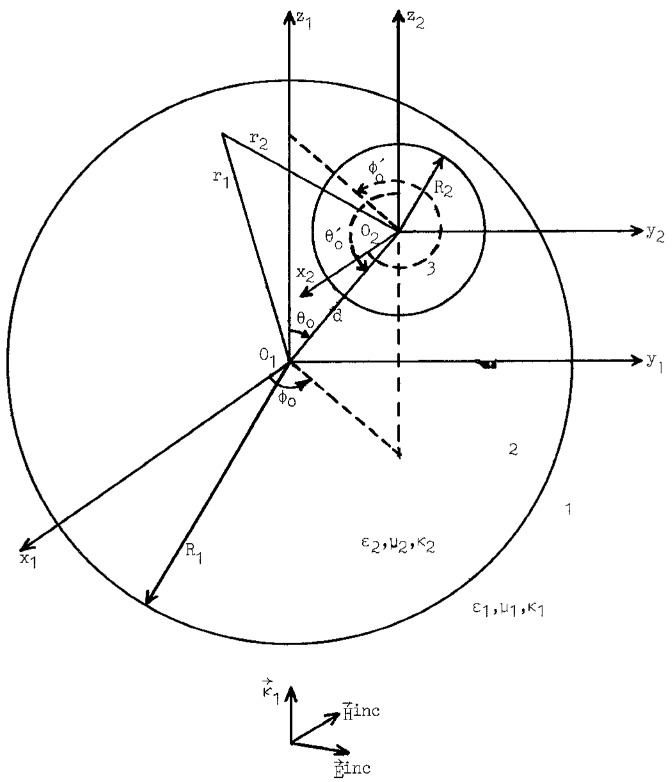

Fig. 1. Geometry of the scatterer.

Let $\vec{E}_2(O), \vec{H}_2(O)$ be the field intensities in region 2, for the unperturbed (homogeneous) dielectric sphere of radius R_1, i.e. in the absence of the inner perfectly conducting sphere. These intensities have the expressions

$$\vec{E}_2(O) = \sum_{n=1}^{\infty} \sum_{m=-1,\,1} [S_{mn}(O)\vec{m}_{mn}^{(1)}(k_2 r_1,\, \theta_1,\, \varphi_1) + T_{mn}(O)\vec{n}_{mn}^{(1)}(k_2 r_1,\, \theta_1,\, \varphi_1)] \quad (3)$$

$$\vec{H}_2(O) = -\frac{\imath}{\zeta_2} \sum_{n=1}^{\infty} \sum_{m=-1,\,1} [S_{mn}(O)\vec{n}_{mn}^{(1)}(k_2 r_1,\, \theta_1,\, \varphi_1) + T_{mn}(O)\vec{m}_{mn}^{(1)}(k_2 r_1,\, \theta_1,\, \varphi_1)], \quad \zeta_2 = (\mu_2/\epsilon_2)^{1/2} \quad (4)$$

The intensities of the scattered field in this case, i.e. in the absence of the inner perfectly conducting sphere, are

$$\vec{E}^{sc}(O) = \sum_{n=1}^{\infty} \sum_{m=-1,\,1} [F_{mn}(O)\vec{m}_{mn}^{(3)}(k_1 r_1,\, \theta_1,\, \varphi_1) + G_{mn}(O)\vec{n}_{mn}^{(3)}(k_1 r_1,\, \theta_1,\, \varphi_1)] \quad (5)$$

$$vecH^{sc}(O) = -\frac{i}{\zeta_1} \sum_{n=1}^{\infty} \sum_{m=-1,\,1} [F_{mn}(O)\vec{n}_{mn}^{(3)}(k_1 r_1,\, \theta_1,\, \varphi_1) + G_{mn}(O)\vec{m}_{mn}^{(3)}(k_1 r_1,\, \theta_1,\, \varphi_1)] \quad (6)$$

The spherical eigenvectors $\vec{m}_{mn}^{(3)}, \vec{n}_{mn}^{(3)}$ are given by the same formulas as $\vec{m}_{mn}^{(1)}, \vec{n}_{mn}^{(1)}$ with the only difference that the spherical Bessel functions of the first kind $j_n(x)$, $j_n^d(x) = d[xj_n(x)]/dx$ are replaced now by the spherical Hankel functions of the first kind $h_n(x)$, $h_n^d(x) = d[xh_n(x)]/dx$, respectively. The superscript (1) is omitted from the Hankel function for simplicity.

The expressions of $S_{mn}(O)$, $T_{mn}(O)$, $F_{mn}(O)$, $G_{mn}(O)$, which are found by the satisfaction of the boundary conditions at $r_1 = R_1$, are given in Appendix A.

It is apparent that the presence of the inner conducting sphere, with small radius, slightly perturbs the above solutions. To determine the modified solutions $\vec{E}_2, \vec{H}_2, \vec{E}^{sc}, \vec{H}^{sc}$, we assume that the standing wave $\vec{E}_2(O), \vec{H}_2(O)$ is incident upon the inner conducting sphere and is scattered by it. Because of the smallness of its radius, the latter scattered field is very weak compared to $\vec{E}_2(O), \vec{H}_2(O)$, so it will slightly perturb the expansion coefficients in (3)–(6). The validity of the above assumption and its limits are examined in Section IV. The modified solutions $\vec{E}_2, \vec{H}_2, \vec{E}^{sc}, \vec{H}^{sc}$ have the following expressions:

$$\vec{E}_2 = \sum_{n=1}^{\infty} \sum_{m=-1,\,1} [S_{mn}\vec{m}_{mn}^{(1)}(k_2 r_1,\, \theta_1,\, \varphi_1) + T_{mn}\vec{n}_{mn}^{(1)}(k_2 r_1,\, \theta_1,\, \varphi_1)] + \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [P_{mn}\vec{m}_{mn}^{(3)}(k_2 r_2,\, \theta_2,\, \varphi_2) + Q_{mn}\vec{n}_{mn}^{(3)}(k_2 r_2,\, \theta_2,\, \varphi_2)] \quad (7)$$

$$\vec{H}_2 = -\frac{i}{\zeta_2}\Bigg\{\sum_{n=1}^{\infty}\sum_{m=-1,1}[S_{mn}\vec{n}_{mn}^{(1)}(k_2r_1, \theta_1, \varphi_1) + T_{mn}\vec{m}_{mn}^{(1)}(k_2r_1, \theta_1, \varphi_1)] + \sum_{n=1}^{\infty}\sum_{m=-n}^{n}[P_{mn}\vec{n}_{mn}^{(3)}(k_2r_2, \theta_2, \varphi_2) + Q_{mn}\vec{m}_{mn}^{(3)}(k_2r_2, \theta_2, \varphi_2)]\Bigg\} \quad (8)$$

$$\vec{E}^{sc} = \sum_{n=1}^{\infty}\sum_{m=-1,1}[F_{mn}\vec{m}_{mn}^{(3)}(k_1r_1, \theta_1, \varphi_1) + G_{mn}\vec{n}_{mn}^{(3)}(k_1r_1, \theta_1, \varphi_1)] \quad (9)$$

$$\vec{H}^{sc} = -\frac{i}{\zeta_1}\sum_{n=1}^{\infty}\sum_{m=-1,1}[F_{mn}\vec{n}_{mn}^{(3)}(k_1r_1, \theta_1, \varphi_1) + G_{mn}\vec{m}_{mn}^{(3)}(k_1r_1, \theta_1, \varphi_1)] \quad (10)$$

In order to satisfy the boundary conditions at $r_1 = R_1$ and $r_2 = R_2$, we re-expand (7) and (8) in terms of the spherical eigenvectors around the origins O_1 and O_2, respectively, using the well-known translational addition theorems [16], [17]:

$$\vec{m}_{mn}^{(1)}(k_2r_1, \theta_1, \varphi_1) = \sum_{s=1}^{\infty}\sum_{\mu=-s}^{s}[A_{\mu s}^{mn}\vec{m}_{\mu s}^{(1)}(k_2r_2, \theta_2, \varphi_2) + B_{\mu s}^{mn}\vec{n}_{\mu s}^{(1)}(k_2r_2, \theta_2, \varphi_2)] \quad (11)$$

$$\vec{m}_{mn}^{(3)}(k_2r_2, \theta_2, \varphi_2) = \sum_{s=1}^{\infty}\sum_{\mu=-s}^{s}[C_{\mu s}^{mn}\vec{m}_{\mu s}^{(3)}(k_2r_1, \theta_1, \varphi_1) + D_{\mu s}^{mn}\vec{n}_{\mu s}^{(3)}(k_2r_1, \theta_1, \varphi_1)], \quad (r_1 > d) \quad (12)$$

with

$$A_{\mu s}^{mn} = (-1)^{\mu}\sum_{p}a(m, n| -\mu, s|p)a(n, s, p)j_p(k_2d) \cdot P_p^{m-\mu}(\cos\theta_0)e^{i(m-\mu)\varphi_0} \quad (13)$$

$$B_{\mu s}^{mn} = (-1)^{\mu}\sum_{p}a(m, n| -\mu, s|p+1, p)b(n, s, p+1) \cdot j_{p+1}(k_2d)P_{p+1}^{m-\mu}(\cos\theta_0)e^{i(m-\mu)\varphi_0} \quad (14)$$

where P_p^q are the associated Legendre functions.

The expansion coefficients $C_{\mu s}^{mn}$ and $D_{\mu s}^{mn}$ appearing in (12) are given by (13) and (14), respectively, with θ_O', φ_O' in place of θ_O, φ_O where d, θ_O', φ_O' are the coordinates of O_1 with respect to O_2. In (13) and (14), the summation index p varies from $|n-s|$ to $n+s$ by steps of 2. From (11) and (12), one can also easily find the addition theorems for $\vec{n}_{mn}^{(1)}(k_2r_1, \theta_1, \varphi_1), \vec{n}_{mn}^{(3)}(k_2r_2, \theta_2, \varphi_2)$, respectively, by simply interchanging $\vec{m}_{\mu s}$ and $\vec{n}_{\mu s}$ inside the summations. For the calculation of the symbols $a(m, n| -\mu, s|p), a(n, s, p)$ etc., see Appendix B.

It can be easily seen from Fig. 1 that in any case $\theta_O + \theta_O' = \pi$ and $\varphi_0 - \varphi_O' = \pm\pi$. So, by using the relations $\cos\theta_O' = -\cos\theta_O$, $\exp[i(m-\mu)\varphi_O'] = (-1)^{m-\mu}\exp[i(m-\mu)\varphi_O]$, $P_n^m(-x) = (-1)^{n+m}P_n^m(x)$, and the fact that the summation index p in (13), (14) varies by steps of 2, one can easily prove the following very simple general relations connecting C and D with A and B, respectively:

$$C_{\mu s}^{mn} = (-1)^{n+s}A_{\mu s}^{mn}, \qquad D_{\mu s}^{mn} = (-1)^{n+s+1}B_{\mu s}^{mn} \quad (15)$$

The unknown expansion coefficients appearing in (7)–(10) are found by the satisfaction of the boundary conditions at $r_1 = R_1$:

$$\hat{r}_1 \times (\vec{E}^{inc} + \vec{E}^{sc}) = \hat{r}_1 \times \vec{E}_2,$$

$$\hat{r}_1 \times (\vec{H}^{inc} + \vec{H}^{sc}) = \hat{r}_1 \times \vec{H}_2 \quad (16)$$

as well as at $r_2 = R_2$

$$\hat{r}_2 \times \vec{E}_2 = 0 \quad (17)$$

Satisfying (17) by the use of (11) (and the analogous for $\vec{n}_{mn}^{(1)}(k_2r_1, \theta_1, \varphi_1)$) into (7) and using next the orthogonal properties of the spherical vector functions $\vec{B}, \vec{C}$ [14] appearing in the expressions of $\vec{m}_{mn}, \vec{n}_{mn}$, we conclude after straightforward but lengthy calculations to the results ($|m| \leqq n, n \geqq 1$)

$$P_{mn} = L_n\sum_{l=1}^{\infty}\sum_{u=-1,1}[A_{mn}^{ul}S_{ul} + B_{mn}^{ul}T_{ul}] \quad (18)$$

$$Q_{mn} = L_n^d\sum_{l=1}^{\infty}\sum_{u=-1,1}[B_{mn}^{ul}S_{ul} + A_{mn}^{ul}T_{ul}] \quad (19)$$

where

$$L_n = -j_n(\rho_2)/h_n(\rho_2), \qquad L_n^d = -j_n^d(\rho_2)/h_n^d(\rho_2), \qquad \rho_2 = k_2R_2 \quad (20)$$

Satisfaction of the boundary conditions (16), by using (12) (and the analogous for $\vec{n}_{mn}^{(3)}(k_2r_2, \theta_2, \varphi_2)$) into (7) and (8) (the restriction $r_1 > d$ is always valid at $r_1 = R_1$), and by following analogous steps to those leading in (18), (19), lead us now, after lengthy calculations, to the following sets of equations ($m = -1, 1, n \geqq 1$):

$$S_{mn}j_n(\rho_3) + h_n(\rho_3)\sum_{l=1}^{\infty}\sum_{u=-l}^{l}[C_{mn}^{ul}P_{ul} + D_{mn}^{ul}Q_{ul}] = F_{mn}h_n(\rho_1) - (E_{ox} - imE_{oy})g_{mn}j_n(\rho_1) \quad (21)$$

$$T_{mn}j_n^d(\rho_3) + h_n^d(\rho_3)\sum_{l=1}^{\infty}\sum_{u=-l}^{l}[C_{mn}^{ul}Q_{ul} + D_{mn}^{ul}P_{ul}] = \frac{k_2}{k_1}[G_{mn}h_n^d(\rho_1) - m(E_{ox} - imE_{oy})g_{mn}j_n^d(\rho_1)] \quad (22)$$

$$S_{mn}j_n^d(\rho_3) + h_n^d(\rho_3)\sum_{l=1}^{\infty}\sum_{u=-l}^{l}[C_{mn}^{ul}P_{ul} + D_{mn}^{ul}Q_{ul}] = \frac{\mu_2}{\mu_1}[F_{mn}h_n^d(\rho_1) - (E_{ox} - imE_{oy})g_{mn}j_n^d(\rho_1)] \quad (23)$$

$$T_{mn}j_n(\rho_3) + h_n(\rho_3)\sum_{l=1}^{\infty}\sum_{u=-l}^{l}[C_{mn}^{ul}Q_{ul} + D_{mn}^{ul}P_{ul}]$$
$$= \frac{k_1\mu_2}{k_2\mu_1}[G_{mn}h_n(\rho_1) - m(E_{ox} - imE_{oy})g_{mn}j_n(\rho_1)] \quad (24)$$

where

$$\rho_1 = k_1R_1, \qquad \rho_3 = k_2R_1, \qquad g_{1n} = i^{n+1}\frac{2n+1}{2n(n+1)},$$
$$g_{-1n} = i^{n+1}\frac{2n+1}{2}. \quad (25)$$

By eliminating the scattering coefficients F_{mn} and G_{mn} from (21)–(24) and by using (18) and (19), two infinite sets of linear nonhomogeneous equations are obtained for the expansion coefficients S_{mn} and T_{mn}:

$$S_{mn} + I_n\sum_{l=1}^{\infty}\sum_{v=1}^{\infty}\sum_{u=-l}^{l}\sum_{w=-1,\,1}\{[L_lC_{mn}^{ul}A_{ul}^{wv} + L_l^dD_{mn}^{ul}B_{ul}^{wv}]$$
$$\cdot S_{wv} + [L_lC_{mn}^{ul}B_{ul}^{wv} + L_l^dD_{mn}^{ul}A_{ul}^{wv}]T_{wv}\}$$
$$= S_{mn}(O), \qquad (m = -1,\, 1, n \geqq 1) \quad (26)$$

$$T_{mn} + I'_n\sum_{l=1}^{\infty}\sum_{v=1}^{\infty}\sum_{u=-l}^{l}\sum_{w=-1,\,1}\{[L_l^dC_{mn}^{ul}B_{ul}^{wv} + L_lD_{mn}^{ul}A_{ul}^{wv}]$$
$$\cdot S_{wv} + [L_l^dC_{mn}^{ul}A_{ul}^{wv} + L_lD_{mn}^{ul}B_{ul}^{wv}]T_{wv}\}$$
$$= T_{mn}(O), \qquad (m = -1,\, 1, n \geqq 1) \quad (27)$$

In (26) and (27), we have made the substitutions

$$I_n = \frac{\mu_1h_n(\rho_1)h_n^d(\rho_3) - \mu_2h_n^d(\rho_1)h_n(\rho_3)}{\mu_1h_n(\rho_1)j_n^d(\rho_3) - \mu_2h_n^d(\rho_1)j_n(\rho_3)} \quad (28)$$
$$I'_n = \frac{\epsilon_1h_n(\rho_1)h_n^d(\rho_3) - \epsilon_2h_n^d(\rho_1)h_n(\rho_3)}{\epsilon_1h_n(\rho_1)j_n^d(\rho_3) - \epsilon_2h_n^d(\rho_1)j_n(\rho_3)}. \quad (29)$$

For general values of ρ_2, the infinite sets of (26) and (27) can be solved numerically by truncating the summations in order to obtain a matrix equation. However, if ρ_2 is small, an analytical procedure can be developed. As $\rho_2 \to O$, we use the following limiting values [18]:

$$j_n(\rho_2) \to \frac{\rho_2^n}{1.3.5\ldots(2n+1)},$$
$$j_n^d(\rho_2) \to \frac{(n+1)\rho_2^n}{1.3.5\ldots(2n+1)},$$
$$h_n(\rho_2) \to -i\frac{1.3.5\ldots(2n-1)}{\rho_2^{n+1}},$$
$$h_n^d(\rho_2) \to i\frac{1.3.5\ldots(2n-1).n}{\rho_2^{n+1}}, \qquad (n \geqq 1). \quad (30)$$

Next, using (30) into (26) and (27), we retain only the dominant (first-order) terms with respect to ρ_2. Keeping in mind that in this case the expansion coefficients S_{mn}, T_{mn}, F_{mn} and G_{mn} can be approximated by the formula

$$V_{mn} = V_{mn}(O) + \delta V_{mn} \quad (31)$$

where V stands for anyone of S, T, F, G and δV_{mn} is a small perturbation of $V_{mn}(O)$, we substitute in (26) and (27), and we finally conclude at the relations $(m = -1,\, 1,\, n \geqq 1)$:

$$\delta S_{mn} = \frac{i\rho_2^3}{3}I_n\sum_{v=1}^{\infty}\sum_{u=-1}^{1}\sum_{w=-1,\,1}\{[C_{mn}^{u1}A_{u1}^{wv} - 2D_{mn}^{u1}B_{u1}^{wv}]$$
$$\cdot S_{wv}(O) + [C_{mn}^{u1}B_{u1}^{wv} - 2D_{mn}^{u1}A_{u1}^{wv}]T_{wv}(O)\} \quad (32)$$

$$\delta T_{mn} = \frac{i\rho_2^3}{3}I'_n\sum_{v=1}^{\infty}\sum_{u=-1}^{1}\sum_{w=-1,\,1}\{[-2C_{mn}^{u1}B_{u1}^{wv} + D_{mn}^{u1}A_{u1}^{wv}]$$
$$\cdot S_{wv}(O) + [-2C_{mn}^{u1}A_{u1}^{wv} + D_{mn}^{u1}B_{u1}^{wv}]T_{wv}(O)\}. \quad (33)$$

In (32) and (33), we have used the limiting values ((20), (30))

$$L_1 = -\frac{i}{3}\rho_2^3, \qquad L_1^d = \frac{2i}{3}\rho_2^3, \qquad L_n,\, L_n^d = O(\rho_2^{2n+1}),$$
$$(n \geqq 2) \quad (34)$$

and we have omitted the higher-order products $L_l\delta S_{wv}$, $L_l^d\delta S_{wv}$, $L_l\delta T_{wv}$, $L_l^d\delta T_{wv}$. It is evident from (26), (27), (34) that the dominant terms are these with $l = 1$. The omitted terms with $l \geqq 2$ are of order ρ_2^5 or higher.

By returning next in (21)–(24) and by eliminating the coefficients P_{ul}, Q_{ul} with the use of (18) and (19), we obtain the expressions for the scattered field coefficients F_{mn} and G_{mn}. More analytically, from (21) or (23) (from (22) or (24)) by using (31), (26) and (A1), (A3) from the Appendix A ((31), (27) and (A2), (A4) from the Appendix A), we finally conclude—after some manipulation—the following relations for $\delta F_{mn}(\delta G_{mn})$:

$$\delta F_{mn} = \left[\frac{j_n(\rho_3)}{h_n(\rho_1)} - \frac{h_n(\rho_3)}{I_nh_n(\rho_1)}\right]\delta S_{mn},$$
$$\delta G_{mn} = \frac{\rho_1}{\rho_3}\left[\frac{j_n^d(\rho_3)}{h_n^d(\rho_1)} - \frac{h_n^d(\rho_3)}{I'_nh_n^d(\rho_1)}\right]\delta T_{mn}. \quad (35)$$

Using next the asymptotic expansion for $h_n(k_1r_1)$ [13] in (9) we take the scattered far field expression

$$\vec{E}_f^{sc} = \frac{e^{ik_1r_1}}{r_1}\vec{f}(\theta_1,\, \varphi_1) \quad (36)$$

in terms of the scattering amplitude

$$\vec{f}(\theta_1,\, \varphi_1) = \sum_{n=1}^{\infty}\sum_{m=-1,\,1}[n(n+1)]^{1/2}\frac{(-i)^{n+1}}{k_1}$$
$$\cdot [F_{mn}\vec{C}_{mn}(\theta_1,\, \varphi_1) + iG_{mn}\vec{B}_{mn}(\theta_1,\, \varphi_1)]$$
$$= f_\theta(\theta_1,\, \varphi_1)\hat{\theta} + f_\varphi(\theta_1,\, \varphi_1)\hat{\varphi}. \quad (37)$$

Setting now

$$\vec{B}_{mn}(\theta_1,\, \varphi_1) = e^{im\varphi_1}[B_{mn}^\theta(\theta_1)\hat{\theta} + B_{mn}^\varphi(\theta_1)\hat{\varphi}],$$
$$\vec{C}_{mn}(\theta_1,\, \varphi_1) = e^{im\varphi_1}[C_{mn}^\theta(\theta_1)\hat{\theta} + C_{mn}^\varphi(\theta_1)\hat{\varphi}] \quad (38)$$

we have from [14] that

$$B^{\theta}_{mn}(\theta_1) = -C^{\varphi}_{mn}(\theta_1), \qquad B^{\varphi}_{mn}(\theta_1) = C^{\theta}_{mn}(\theta_1),$$

$$B^{\theta^*}_{mn}(\theta_1) = B^{\theta}_{mn}(\theta_1), \qquad B^{\varphi^*}_{mn}(\theta_1) = -B^{\varphi}_{mn}(\theta_1),$$

$$(m = -1, 1), \qquad B^{\theta}_{1n}(\theta_1) = -n(n+1)B^{\theta}_{-1n}(\theta_1),$$

$$B^{\varphi}_{1n}(\theta_1) = n(n+1)B^{\varphi}_{-1n}(\theta_1) \tag{39}$$

In (37)–(39) $\hat{\theta}$, $\hat{\varphi}$ are spherical unit vectors and the star (*) indicates the conjugate complex number.

The differential scattering cross-section is defined as follows:

$$\sigma(\theta_1, \varphi_1) = \left|\vec{f}(\theta_1, \varphi_1)\right|^2 = |f_\theta(\theta_1, \varphi_1)|^2 + |f_\varphi(\theta_1, \varphi_1)|^2 \tag{40}$$

where from (37)–(39):

$$f_\theta(\theta_1, \varphi_1) = \sum_{n=1}^{\infty}\sum_{m=-1,1}[n(n+1)]^{1/2}\frac{(-i)^{n+1}}{k_1} \cdot [|q_{mn}|F_{mn}B^{\varphi}_{1n}(\theta_1) + q_{mn}G_{mn}B^{\theta}_{1n}(\theta_1)]e^{im\varphi_1} \tag{41}$$

$$f_\varphi(\theta_1, \varphi_1) = \sum_{n=1}^{\infty}\sum_{m=-1,1}[n(n+1)]^{1/2}\frac{(-i)^{n}}{k_1} \cdot [q_{mn}F_{mn}B^{\theta}_{1n}(\theta_1) + |q_{mn}|G_{mn}B^{\varphi}_{1n}(\theta_1)]e^{im\varphi_1} \tag{42}$$

and

$$q_{1n} = i, \qquad q_{-1n} = -\frac{i}{n(n+1)}. \tag{43}$$

The backscattering (radar), forward and total scattering cross-sections are, respectively

$$\sigma_b = 4\pi\sigma(\pi, \varphi_1), \qquad \sigma_f = 4\pi\sigma(0, \varphi_1),$$
$$Q_t = \int_{\theta_1=0}^{\pi}\int_{\varphi_1=0}^{2\pi}\sigma(\theta_1, \varphi_1)\sin\theta_1\, d\theta_1\, d\varphi_1. \tag{44}$$

Substituting from (31) into (41) and (42), we obtain

$$f_\theta(\theta_1, \varphi_1) = f^0_\theta(\theta_1, \varphi_1) + \delta f_\theta(\theta_1, \varphi_1),$$
$$f_\varphi(\theta_1, \varphi_1) = f^0_\varphi(\theta_1, \varphi_1) + \delta f_\varphi(\theta_1, \varphi_1) \tag{45}$$

where f^0_θ, f^0_φ are given again by (41), (42), respectively, with $F_{mn}(O), G_{mn}(O)$ in place of F_{mn}, G_{mn}, while δf_θ, δf_φ are given by the same equations, with $\delta F_{mn}, \delta G_{mn}$ in place of F_{mn}, G_{mn}, respectively.

Next, substituting from (45) into (40), we find

$$\sigma(\theta_1, \varphi_1) = \sigma^0(\theta_1, \varphi_1) + \delta\sigma(\theta_1, \varphi_1) \tag{46}$$

where

$$\sigma^0(\theta_1, \varphi_1) = \left|f^0_\theta(\theta_1, \varphi_1)\right|^2 + \left|f^0_\varphi(\theta_1, \varphi_1)\right|^2 \tag{47}$$

$$\begin{aligned}\delta\sigma(\theta_1, \varphi_1) = 2\{&\mathrm{Re}\,[f^0_\theta(\theta_1, \varphi_1)]\,\mathrm{Re}\,[\delta f_\theta(\theta_1, \varphi_1)] \\ &+ \mathrm{Im}\,[f^0_\theta(\theta_1, \varphi_1)]\,\mathrm{Im}\,[\delta f_\theta(\theta_1, \varphi_1)] \\ &+ \mathrm{Re}\,[f^0_\varphi(\theta_1, \varphi_1)]\mathrm{Re}\,[\delta f_\varphi(\theta_1, \varphi_1)] \\ &+ \mathrm{Im}\,[f^0_\varphi(\theta_1, \varphi_1)]\,\mathrm{Im}\,[\delta f_\varphi(\theta_1, \varphi_1)]\}.\end{aligned} \tag{48}$$

In (48), we have omitted second order differentials. Also, the operator Re (Im) represents the Real (Imaginary) part. From (44), (46)–(48) we obtain easily the zero order (unperturebed) term, as well as its first-order perturbation $\delta\sigma$ for the various scattering cross-sections, due to the presence of the metallic sphere with small radius. It can be proved analytically after straightforward but lengthy calculations, the expected result that σ_b and σ_f are independent of φ_1.

Finally, from (44) and (46) we find

$$Q_t = Q^0_t + \delta Q_t \tag{49}$$

where

$$Q^0_t = \frac{8\pi}{k_1^2}\sum_{n=1}^{\infty}\frac{n^2(n+1)^2}{2n+1}[|F_{1n}(O)|^2 + |G_{1n}(O)|^2] \tag{50}$$

$$\begin{aligned}\delta Q_t = \frac{8\pi}{k_1^2}\sum_{n=1}^{\infty}\sum_{m=-1,1}&\frac{n(n+1)}{2n+1} \\ \cdot \{&(E^2_{ox} - E^2_{oy})[\mathrm{Re}\,(F_{mn}(O))\,\mathrm{Re}\,(\delta F_{-mn}) \\ &+ \mathrm{Im}\,(F_{mn}(O))\,\mathrm{Im}\,(\delta F_{-mn}) \\ &- \mathrm{Re}\,(G_{mn}(O))\,\mathrm{Re}\,(\delta G_{-mn}) \\ &- \mathrm{Im}\,(G_{mn}(O))\,\mathrm{Im}\,(\delta G_{-mn})] \\ &+ 2mE_{ox}E_{oy}[\mathrm{Re}\,(F_{mn}(O))\,\mathrm{Im}\,(\delta F_{-mn}) \\ &- \mathrm{Im}\,(F_{mn}(O))\,\mathrm{Re}\,(\delta F_{-mn}) \\ &- \mathrm{Re}\,(G_{mn}(O))\,\mathrm{Im}\,(\delta G_{-mn}) \\ &+ \mathrm{Im}\,(G_{mn}(O))\,\mathrm{Re}\,(\delta G_{-mn})]\}.\end{aligned} \tag{51}$$

The results (50) and (51) were obtained after very lengthy, but straightforward, calculations by substituting from (41) and (42), (46)–(48) in the last of (44) and by using the expressions for the B's in (41) and (42) in terms of Legendre functions [14], as well as some known integrals of these last functions [13]. These results should conform with the forward scattering theorem [19], which for the present problem has the expression

$$Q_t = \frac{4\pi}{k_1}\,\mathrm{Im}\,[\vec{E}_0 \cdot \vec{f}(0, 0)]. \tag{52}$$

Using the fact that for $\theta_1 = \varphi_1 = 0, \hat{x} = \hat{\theta}$ and $\hat{y} = \hat{\varphi}$, we have

$$Q_t = \frac{4\pi}{k_1}\,\mathrm{Im}\,[E_{ox}f_\theta(0, 0) + E_{oy}f_\varphi(0, 0)] \tag{53}$$

where f_θ, f_φ are given in (41) and (42). Substituting from (45), (49) in (53) we obtain

$$Q^0_t = \frac{4\pi}{k_1}\,\mathrm{Im}\,[E_{ox}f^0_\theta(0, 0) + E_{oy}f^0_\varphi(0, 0)],$$
$$\delta Q_t = \frac{4\pi}{k_1}\,\mathrm{Im}\,[E_{ox}\delta f_\theta + E_{oy}\delta f_\varphi]. \tag{54}$$

The very lengthy expressions of the various coefficients do not permit the analytical proof of (54). Their validity was verified numerically to a very good accuracy, for all values of the parameters that were used, providing a very good check for the correctness of our solution.

III. Dielectric Sphere of Small Radius, Embedded Into Another Dielectric One

In the present section, we examine the scattering of a plane electromagnetic wave by a dielectric sphere of small radius, coated eccentrically by another dielectric one. The geometry of the problem is again shown in Fig. 1. The only difference from Section II is that the perfectly conducting sphere of region 3 is now substituted by a lossless dielectric one, with parameters $\epsilon_3,\ \mu_3,\ k_3$.

The fields in regions 1 and 2 are again given by (1), (2), (9), (10) and (7) and (8), respectively. Of course, the various expansion coefficients have different expressions, which will be given here. In the present problem, there is also an electromagnetic field inside the inner dielectric sphere. The intensities of this last field are

$$\vec{E}_3 = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} [X_{mn}\vec{m}_{mn}^{(1)}(k_3r_2,\theta_2,\varphi_2) + Z_{mn}\vec{n}_{mn}^{(1)}(k_3r_2,\theta_2,\varphi_2)] \quad (55)$$

$$\vec{H}_3 = -\frac{i}{\zeta_3}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} [X_{mn}\vec{n}_{mn}^{(1)}(k_3r_2,\theta_2,\varphi_2) + Z_{mn}\vec{m}_{mn}^{(1)}(k_3r_2,\theta_2,\varphi_2)],\qquad \zeta_3=(\mu_3/\epsilon_3)^{1/2} \quad (56)$$

where $\vec{m}_{mn}^{(1)}, \vec{n}_{mn}^{(1)}$ are now expressed with respect to the origin O_2. The boundary conditions at $r_1 = R_1$ are again given by (16), but these at $r_2 = R_2$ are in this case the following:

$$\hat{r}_2 \times (\vec{E}_2 - \vec{E}_3) = 0, \qquad \hat{r}_2 x(\vec{H}_2 - \vec{H}_3) = 0. \quad (57)$$

Satisfaction of the boundary conditions (57) by the use of (11) (and the analogous for $\vec{n}_{mn}^{(1)}(k_2r_1,\theta_1,\varphi_1)$) into (7) and (8) and the orthogonal properties of the vector functions $\vec{B}, \vec{C}$ concludes finally to the following sets of equations ($|m| \leqq n, n \geqq 1$):

$$j_n(\rho_2)\sum_{l=1}^{\infty}\sum_{u=-1,1}[A_{mn}^{ul}S_{ul}+B_{mn}^{ul}T_{ul}] + P_{mn}h_n(\rho_2) = X_{mn}j_n(\rho_4), \qquad (\rho_4 = k_3R_2) \quad (58)$$

$$j_n^d(\rho_2)\sum_{l=1}^{\infty}\sum_{u=-1,1}[B_{mn}^{ul}S_{ul}+A_{mn}^{ul}T_{ul}] + Q_{mn}h_n^d(\rho_2) = \frac{k_2}{k_3}Z_{mn}j_n^d(\rho_4) \quad (59)$$

$$j_n^d(\rho_2)\sum_{l=1}^{\infty}\sum_{u=-1,1}[A_{mn}^{ul}S_{ul}+B_{mn}^{ul}T_{ul}] + P_{mn}h_n^d(\rho_2) = \frac{\mu_2}{\mu_3}X_{mn}j_n^d(\rho_4) \quad (60)$$

$$j_n(\rho_2)\sum_{l=1}^{\infty}\sum_{u=-1,1}[B_{mn}^{ul}S_{ul}+A_{mn}^{ul}T_{ul}] + Q_{mn}h_n(\rho_2) = \frac{k_3\mu_2}{k_2\mu_3}Z_{mn}j_n(\rho_4). \quad (61)$$

By eliminating the coefficients X_{mn}, Z_{mn} from (58)–(61), we obtain again (18) and (19) with the only difference in the expressions of the coefficients L_n and L_n^d, which in this case are

$$L_n = \frac{\mu_2 j_n(\rho_2)j_n^d(\rho_4) - \mu_3 j_n^d(\rho_2)j_n(\rho_4)}{\mu_3 h_n^d(\rho_2)j_n(\rho_4) - \mu_2 h_n(\rho_2)j_n^d(\rho_4)},$$
$$L_n^d = \frac{\epsilon_2 j_n(\rho_2)j_n^d(\rho_4) - \epsilon_3 j_n^d(\rho_2)j_n(\rho_4)}{\epsilon_3 h_n^d(\rho_2)j_n(\rho_4) - \epsilon_2 h_n(\rho_2)j_n^d(\rho_4)}. \quad (62)$$

Satisfaction of the boundary conditions (16), provides again (21)–(25), with the only difference in the expressions of P_{ul}, Q_{ul}, caused by the new L_l, L_l^d, respectively, given in (62). Following next the same steps as in Section II, we conclude to (26)–(29). Then, we use the limiting values (30) into (26), (27) and retain only the dominant terms with respect to $\rho_2(\rho_4 = \rho_2k_3/k_2)$. By substituting there from (31), we finally find in analogy to (32), (33) the relations ($m = -1, 1, n \geqq 1$)

$$\delta S_{mn} = \frac{2i\rho_2^3}{3}I_n\sum_{v=1}^{\infty}\sum_{u=-1}^{1}\sum_{w=-1,1}\{[q_\mu C_{mn}^{u1}A_{u1}^{wv} + q_\epsilon D_{mn}^{u1}B_{u1}^{wv}]S_{wv}(O) + [q_\mu C_{mn}^{u1}B_{u1}^{wv} + q_\epsilon D_{mn}^{u1}A_{u1}^{wv}]T_{wv}(O)\} \quad (63)$$

$$\delta T_{mn} = \frac{2i\rho_2^3}{3}I_n'\sum_{v=1}^{\infty}\sum_{u=-1}^{1}\sum_{w=-1,1}\{[q_\epsilon C_{mn}^{u1}B_{u1}^{wv} + q_\mu D_{mn}^{u1}A_{u1}^{wv}]S_{wv}(O) + [q_\epsilon C_{mn}^{u1}A_{u1}^{wv} + q_\mu D_{mn}^{u1}B_{u1}^{wv}]T_{wv}(O)\} \quad (64)$$

where

$$q_\epsilon = \frac{\epsilon_2 - \epsilon_3}{2\epsilon_2 + \epsilon_3}, \qquad q_\mu = \frac{\mu_2 - \mu_3}{2\mu_2 + \mu_3}. \quad (65)$$

In (63), (64) we have used the limiting values ((62), (30))

$$L_1 = -\frac{2i\rho_2^3}{3}q_\mu, \qquad L_1^d = -\frac{2i\rho_2^3}{3}q_\epsilon,$$
$$L_n,\ L_n^d = O(\rho_2^{2n+1}), \qquad (n \geqq 2) \quad (66)$$

and we have omitted the higher-order products referred after (34). Also, in this case the dominant terms are the ones with $l = 1$, of order ρ_2^3. The omitted terms are of order ρ_2^5 and higher ($l \geqq 2$).

Using next the procedure described in Section II, we conclude finally to formulas (35) for $\delta F_{mn}, \delta G_{mn}$ and to formulas (36)–(54) for the scattered field and the various scattering cross-sections.

Another check for the correctness of our results, moreover to (54), is to try obtaining the results of Section II from the corresponding ones of the present section by using the limiting values $\epsilon_3 \to \infty$, $\mu_3 \to O$ for the material of region 3, which correspond to a perfect conductor [20]. It is evident from (62) that in this special case L_n and L_n^d tend to the expressions (20). Also, by (65), (66) we see that L_1, L_1^d tend to the expressions (34) ($q_\epsilon \to -1, q_\mu \to 1/2$). The same is true for $\delta S_{mn}, \delta T_{mn}$ in (63), (64) that tend to the corresponding ones in (32), (33).

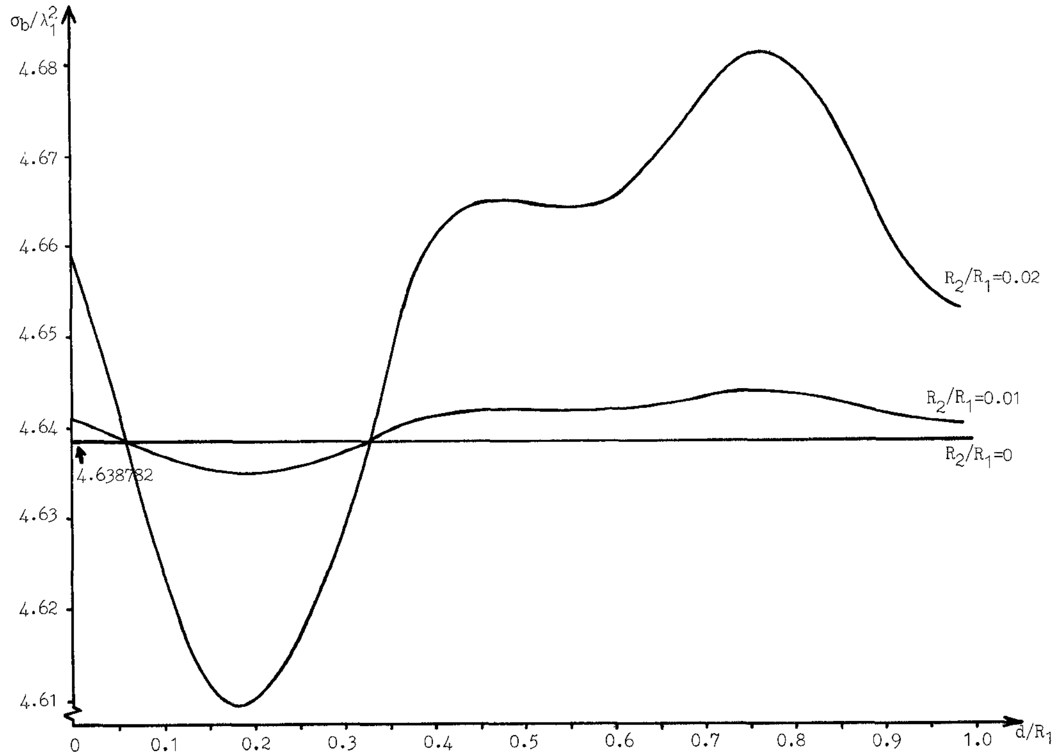

Fig. 2. Backscattering cross-section for $\epsilon_2/\epsilon_1 = 5.5$, $R_1/\lambda_1 = 0.7$, $\theta_o = 45°$, $\phi_o = 45°$, $E_{ox} = 1$, $E_{oy} = 0$ (metallic inner sphere).

Setting now $\epsilon_3 = \epsilon_2$, $\mu_3 = \mu_2(\rho_2 = \rho_4)$ in (62)–(65), we obtain $L_n = L_n^d = 0$, $\delta S_{mn} = \delta T_{mn} = 0$ (also $\delta F_{mn} = \delta G_{mn} = 0$ from (35)), so there results that $P_{mn} = Q_{mn} = 0$ ((18), (19)) and $S_{mn} = S_{mn}(O), T_{mn} = T_{mn}(O), F_{mn} = F_{mn}(O), G_{mn} = G_{mn}(O)$ ((31)) as expected for the problem of a homogeneous dielectric sphere.

Setting next $\epsilon_2 = \epsilon_1$, $\mu_2 = \mu_1(\rho_3 = \rho_1)$ in (26)–(29), we obtain $I_n = I_n' = 0, S_{mn} = S_{mn}(O), T_{mn} = T_{mn}(O)$ ($\delta S_{mn} = \delta T_{mn} = 0$ from (32), (33), (63), (64)), $\vec{E}_2(O) = \vec{E}^{inc}, \vec{H}_2(O) = \vec{H}^{inc}, F_{mn}(O) = G_{mn}(O) = 0$ ((A3), (A4) of Appendix A), $\sigma^o = 0, Q_t^o = 0$ ((41), (42), (47), (50)). In addition, we obtain that δF_{mn} and δG_{mn} are different than zero ((35), (32), (33)), expressing the coefficients of the field scattered by the small inner sphere. Nevertheless, $\delta\sigma$ and δQ_t ((48) and (51)) become equal to zero in this special case where the external dielectric sphere is absent (the zero-order terms vanish) and our assumptions are not valid, so that higher-order terms should also be retained in our solution.

In the special case $d = 0$ (concentric spheres), using the values [18] $j_0(0) = 1, j_n(0) = 0 (n \geq 1)$ we can easily find the various simplified expressions. For example, $P_{mn} = L_n S_{mn}, Q_{mn} = L_n^d T_{mn}$ in (18), (19), because in this special case $A_{mn}^{ul} = 0$ for $u \neq m$ or for $u = m$ and $l \neq n, B_{mn}^{ul} = 0$ for each value of u, l as can be seen by (13), (14) and $A_{mn}^{mn} = 1$, as it is evident from (11) in this case that $r_1 = r_2$, $\theta_1 = \theta_2$ and $\varphi_1 = \varphi_2$. Analogous remarks are valid for C_{mn}^{ul}, D_{mn}^{ul} ((15) or (12)), so the various summations disappear in (21)–(24), substituted only by $P_{mn}, Q_{mn}, P_{mn}, Q_{mn}$, respectively, these in (26), (27) are substituted by $L_n S_{mn}, L_n^d T_{mn}$, respectively, etc. The same results were also obtained from the independent solution of the problem with two concentric spheres for general values of ρ_2, as well as for ρ_2 small.

IV. Numerical Results and Discussion

In Figs. 2–12, the various scattering cross-sections are given for the configuration of Fig. 1 for metallic and dielectric inner sphere. Figs. 2–7 are referred to a metallic inner sphere. More analytically, in Figs. 2 and 3 the backscattering cross-section σ_b is plotted versus d/R_1 and θ_0, respectively, for $E_{ox} = 1, E_{oy} = 0$. In Fig. 4, Q_t (total scattering cross-section), is plotted versus θ_0, for the same polarization. In Fig. 5, σ_b is plotted versus φ_0 for $E_{ox} = 0, E_{oy} = 1$, while in Figs. 6 and 7, σ_b and σ_f (forward scattering cross-section), respectively, are plotted versus d/R_1, for $E_{ox} = E_{oy} = 1/\sqrt{2}$.

Figs. 8–12 are referred to a dielectric inner sphere. In Fig. 8, σ_b is plotted versus θ_0, for $E_{ox} = 1, E_{oy} = 0$. In Fig. 9, σ_b is plotted versus d/R_1, while in Fig. 10, σ_f is plotted versus φ_0, both for $E_{ox} = 0, E_{oy} = 1$.

Finally, in Figs. 11 and 12 σ_b and Q_t, respectively, are plotted versus d/R_1 for $E_{ox} = E_{oy} = 1/\sqrt{2}$.

Our results are symmetrical about the plane defined by the $O_1 z_1$ axis and the direction of polarization $\vec{E}_0$, as well as about its perpendicular one, as it is imposed by the geometry of the scatterer (Figs. 5, 9, 10, and others).

In all figures we have taken $\mu_1 = \mu_2 = \mu_3$. Also, $\lambda_1 = 2\pi/k_1$ is the wavelength in region 1. The case $R_2/R_1 = 0$ corresponds to the problem of the unperturbed (homogeneous) dielectric sphere of radius R_1.

A result expected to hold from reciprocity and seen in Figs. 7, 10 (4, 12) is that $\sigma_f(Q_t)$ has the same values for angles θ_0 with sum equal to π, i.e. for θ_0 and $\pi - \theta_0$.

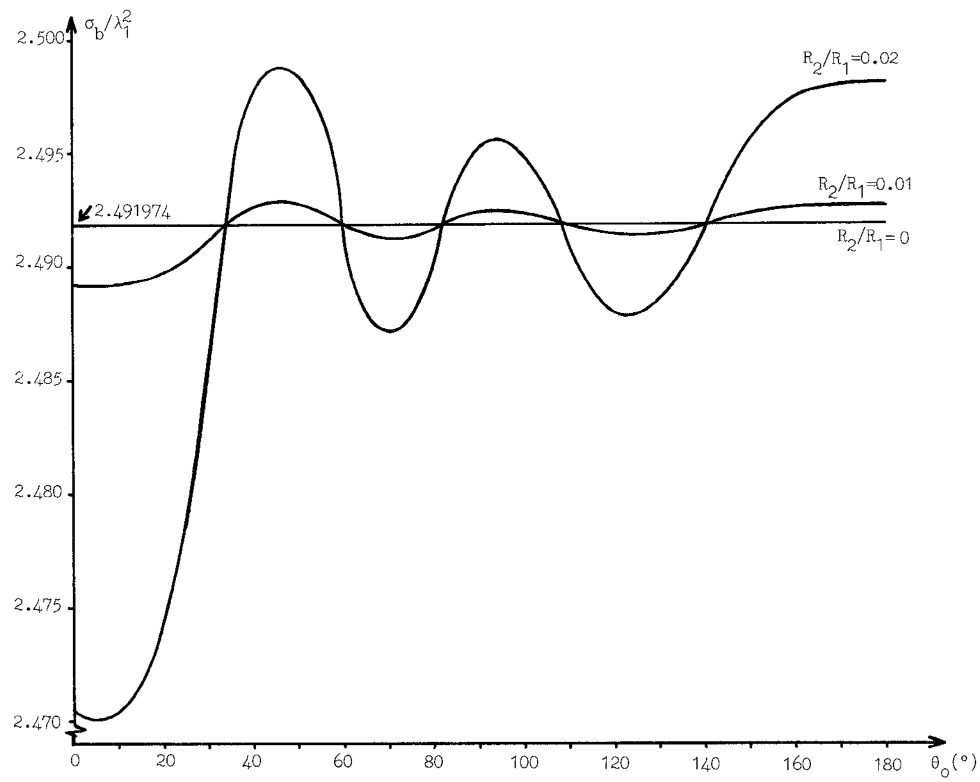

Fig. 3. Backscattering cross-section for $\epsilon_2/\epsilon_1 = 2.54$, $R_1/\lambda_1 = 0\ 7$, $d/R_1 = 0.6$, $\phi_o = 45°$, $E_{ox} = 1$, $E_{oy} = 0$ (metallic inner sphere).

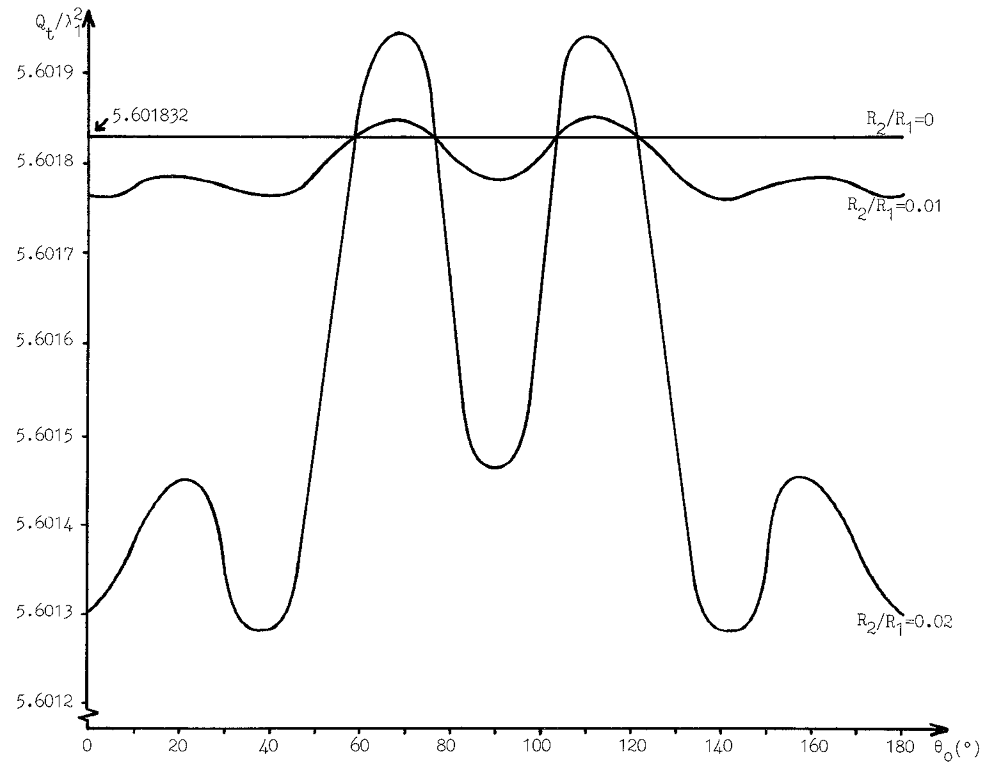

Fig. 4. Total scattering cross-section for $\epsilon_2/\epsilon_1 = 2.54$, $R_1/\lambda_1 = 0.7$, $d/R_1 = 0.6$, $\phi_o = 45°$, $E_{ox} = 1$, $E_{oy} = 0$ (metallic inner sphere).

From the above figures, it is evident that the presence of a metallic or dielectric small sphere inside a dielectric one changes its various scattering cross-sections, increasing or decreasing them depending on the values of the parameters. This change may be useful in practice, to obtain information about inhomogeneities or nonsymmetries inside dielectric spheres by observing their scattered field produced by the incidence of a plane electromagnetic wave. Inversely, one can change the various scattering cross-sections of a dielectric sphere by simply placing a metallic or dielectric small sphere inside it.

More analytically, from Figs. 6, 11, and others available, it seems that σ_b takes its greatest or smallest value or both (depending on the parameters), for $\theta_O = 0$ (small inner sphere

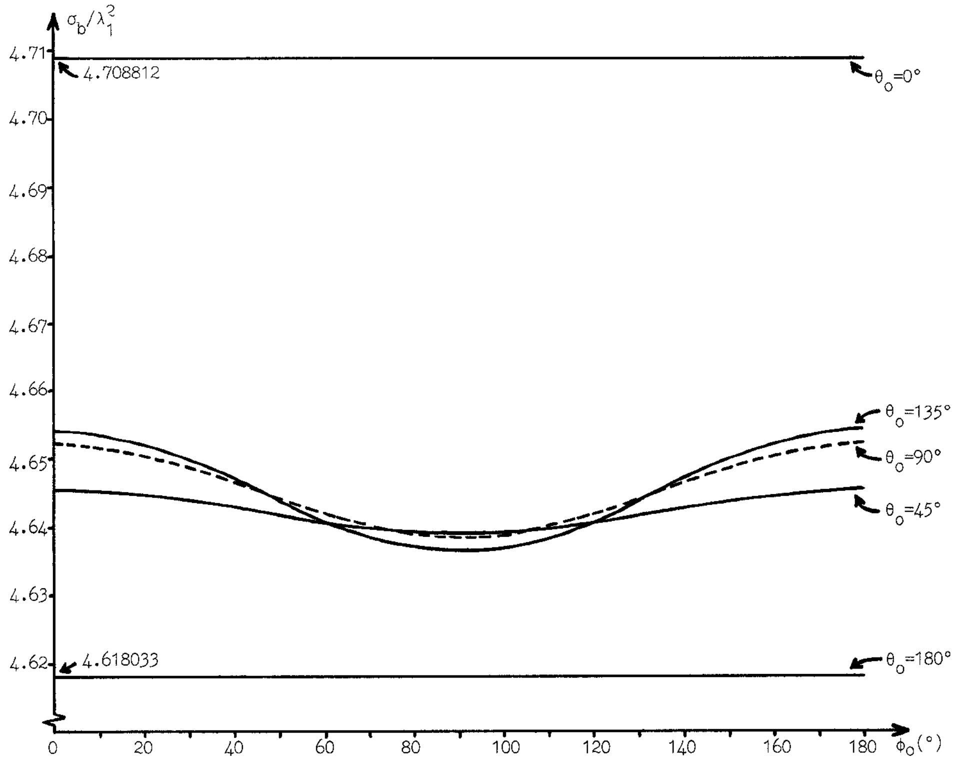

Fig. 5. Backscattering cross-section for $\epsilon_2/\epsilon_1 = 5.5$, $R_1/\lambda_1 = 0.7$, $d/R_1 = 0.6$, $R_2/R_1 = 0.01$, $E_{ox} = 0$, $E_{oy} = 1$ (metallic inner sphere).

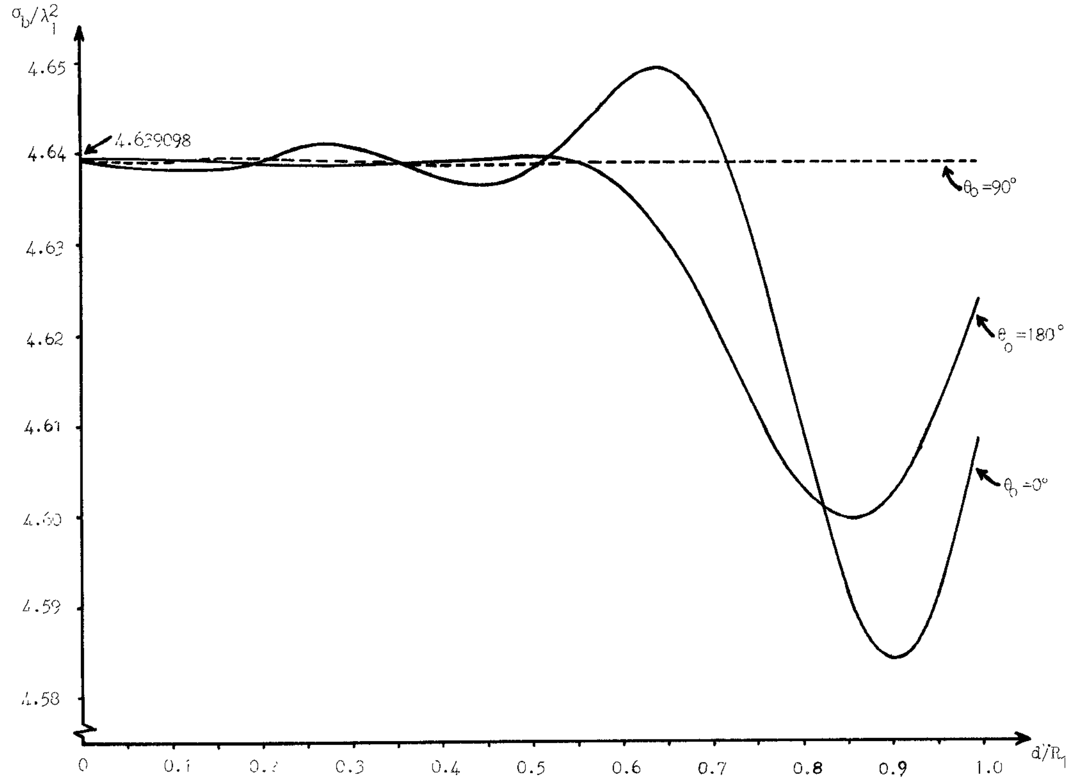

Fig. 6. Backscattering cross-section for $\epsilon_2/\epsilon_1 = 5.5$, $R_1/\lambda_1 = 0.7$, $R_2/R_1 = 0.005$, $\phi_o = 45°$, $E_{ox} = E_{oy} = 1/\sqrt{2}$ (metallic inner sphere).

on O_1z_1 axis) and for some value of d/R_1 in the range $0.5 < d/R_1 < 1$, for metallic as well as for dielectric small inner sphere. This is also valid for σ_f and Q_t, as can be seen from Figs. 7 and 12 (the values for $\theta_O = 0$ and $\theta_O = \pi$ are equal in this case, because of the reciprocity). From the same Figs. 6, 7, 11, 12, and others available, is evident the smaller variation of σ_b, σ_f and Q_t, with respect to d/R_1, in the range $0.5 < d/R_1 < 1$, for intermediate values of $\theta_O (0 < \theta_O < \pi)$,

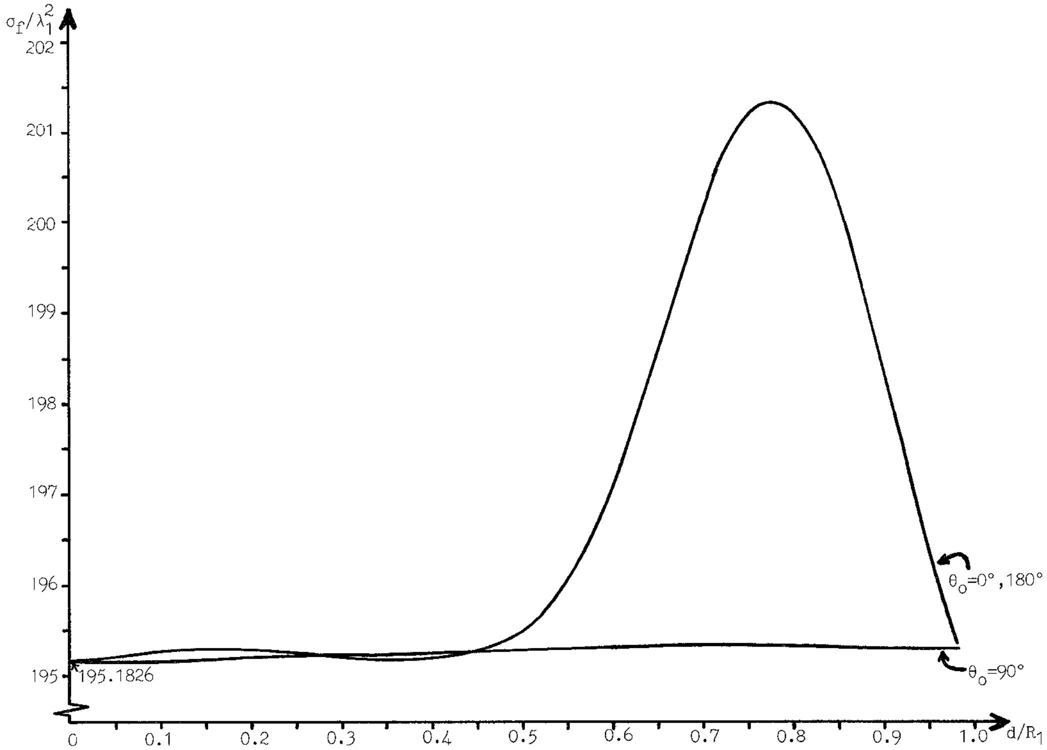

Fig. 7. Forward scattering cross-section for $\epsilon_2/\epsilon_1 = 5.5$, $R_1/\lambda_1 = 0.7$, $R_2/R_1 = 0.02$, $\phi_o = 45°$, $E_{ox} = E_{oy} = 1/\sqrt{2}$ (metallic inner sphere).

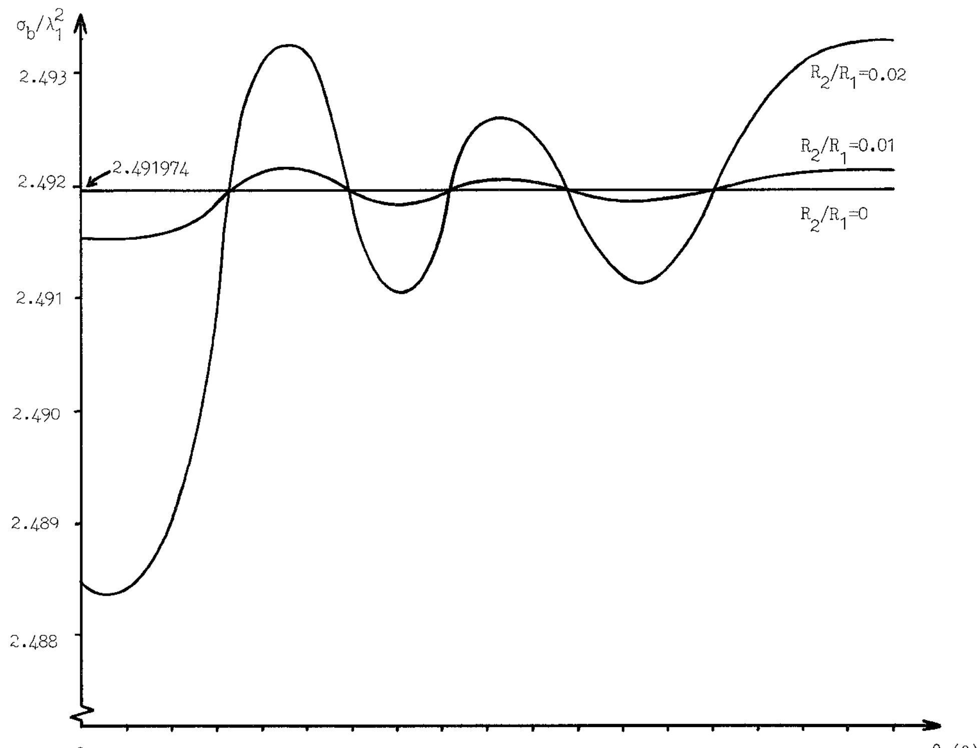

Fig. 8. Backscattering cross-section for $\epsilon_2/\epsilon_1 = 2.54$, $\epsilon_3/\epsilon_1 = 5.5$, $R_1/\lambda_1 = 0.7$, $d/R_1 = 0.6$, $\varphi_o = 45°$, $E_{ox} = 1$, $E_{oy} = 0$ (dielectric inner sphere).

compared to that for $\theta_O = 0$ or $\theta_O = \pi$ (smallest variation for $\theta_O = \pi/2$).

From the figures presented here and from others available, is evident the higher sensitivity of σ_b to internal inhomogeneities, in comparison to that of σ_f and Q_t, especially in the case of metallic ones.

Finally, from Figs. 5, 9, 10, and others available, it is seen that σ_b, σ_f and Q_t appear a maximum or minimum value

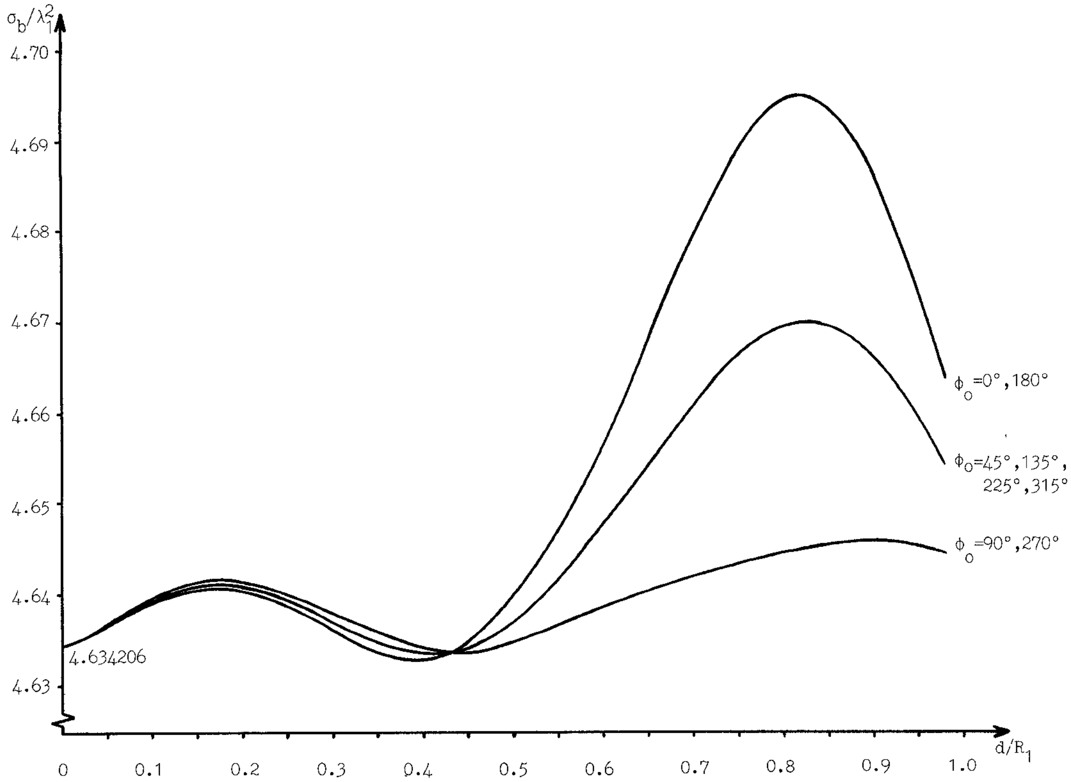

Fig. 9. Backscattering cross-section for $\epsilon_2/\epsilon_1 = 5.5, \epsilon_3/\epsilon_1 = 2.54, R_1/\lambda_1 = 0.7, R_2/R_1 = 0.02, \theta_o = 45°, E_{ox} = 0, E_{oy} = 1$ (dielectric inner sphere).

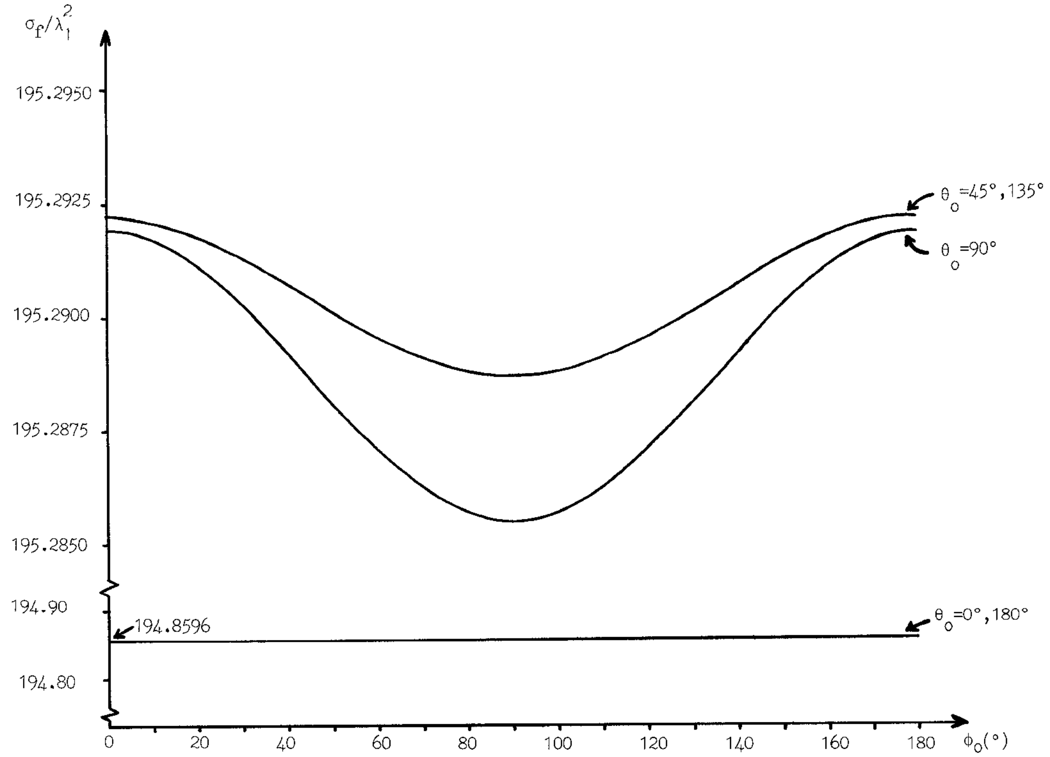

Fig. 10. Forward scattering cross-section for $\epsilon_2/\epsilon_1 = 5.5$, $\epsilon_3/\epsilon_1 = 2.54$, $R_1/\lambda_1 = 0.7$, $d/R_1 = 0.6$, $R_2/R_1 = 0.02$, $E_{ox} = 0$, $E_{oy} = 1$ (dielectric inner sphere).

(depending on the parameters) with respect to φ_O, when φ_O is equal or differs by π rads from the angle between O_1x_1 axis and $\vec{E}_O$, and inversely a minimum or maximum value, when φ_O differs by $\pi/2$ or $3\pi/2$ rads from that angle.

The former observations are very useful for the detection of the small inhomogeneity, i.e. the determination of its position, from the measurement of the scattered field. This position will be on the diameter of the external sphere lying in that

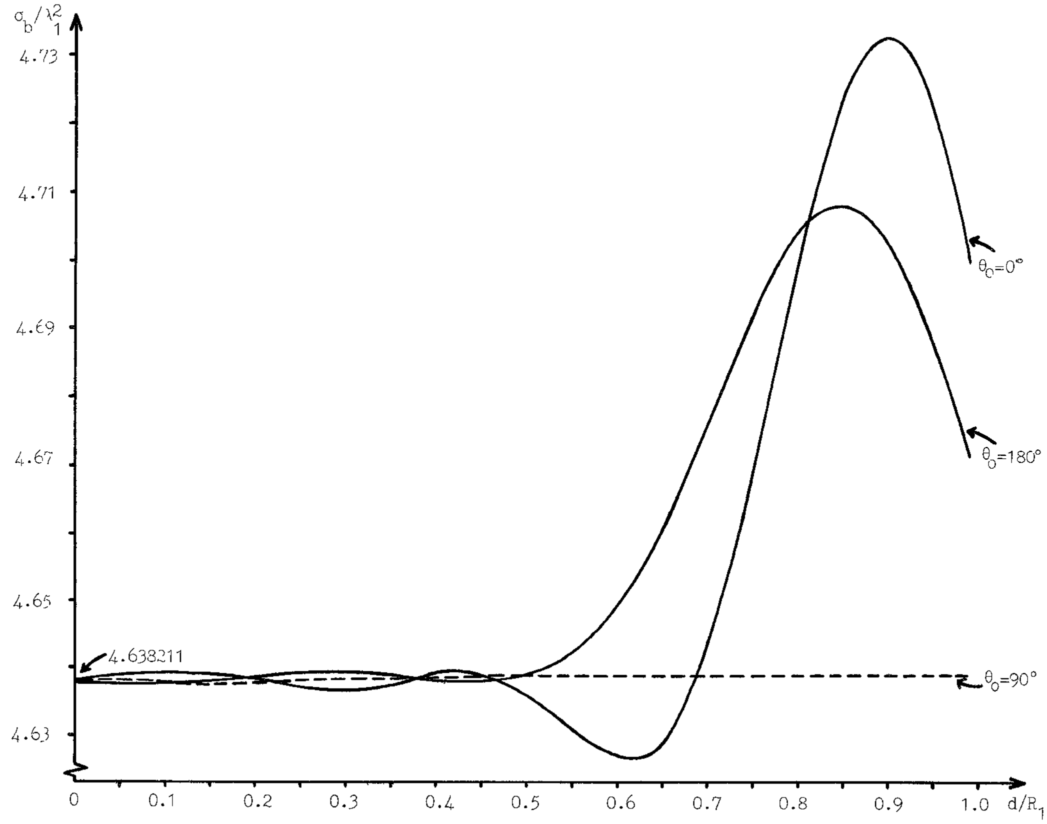

Fig. 11. Backscattering cross-section for $\epsilon_2/\epsilon_1 = 5.5$, $\epsilon_3/\epsilon_1 = 2.54$, $R_1/\lambda_1 = 0.7$, $R_2/R_1 = 0.01$, $\phi_o = 45°$, $E_{ox} = E_{oy} = 1/\sqrt{2}$ (dielectric inner sphere).

direction of propagation of the incident field, which makes the various scattering cross-sections largest or smallest. Moreover, the extremum of σ_b reveals not only the above diameter, but also its radius where the inhomogeneity will lie. This radius is in the opposite side from that of the direction of incidence. The small inhomogeneity will lie also on that plane (or its perpendicular one) defined by the direction of propagation of the incident field and the vector $\vec{E}_O$, where σ_b, σ_f and Q_t appear an extremum.

As it is evident by formulas (32), (33), (63), (64), and (35)–(51), we can easily calculate the various scattering cross-sections for each small value of R_2/R_1, if the rest of the known parameters remain constant, by simply using the results given in Figs. 2–12, because $\delta\sigma$ and δQ_t are analogous to ρ_2^3, i.e. analogous to $(R_2/R_1)^3$. The same is valid for various values of $q_\epsilon \neq 0$ when $q_\mu = 0$ [$q_\mu \neq 0$ when $q_\epsilon = 0$], as it is seen by formulas (63)–(65) and (35)–(51). In this case, δS_{mn}, $\delta T_{mn} \sim q_\epsilon[\sim q_\mu]$ and finally $\delta\sigma$ and δQ_t are analogous to these quantities, so it is easy to calculate the various scattering cross-sections for each different value of $\epsilon_3/\epsilon_2[\mu_3/\mu_2]$ (only of ϵ_3/ϵ_2 for Figs. 2–12 where $\mu_3 = \mu_2$) if the rest known parameters of the problem remain constant. For this purpose, the following values are necessary, which are not marked on the corresponding figures: $\sigma_b^o/\lambda_1^2 = 4.638782$ (Figs. 5, 6, 9, and 11), $\sigma_f^o/\lambda_1^2 = 195.2905$ (Figs. 7 and 10) and $Q_t^o/\lambda_1^2 = 7.71444$ (Fig. 12). Inversely, the above remarks are useful for the calculation of the radius R_2 or of the constitutive parameters of the inner small sphere, i.e. for its identification, from the measurement of the scattered field.

An indication about the range of validity of the assumption referred above (7) and the error bounds of our approximation $\rho_2 \to 0$, can be given in the special case $d = 0$ by comparing our results with the available exact ones for the problem of two concentric spheres. The percent errors of our results, for special values of the parameters, are given in Table I. From this table, it is seen that, at least in this special case, these errors are low enough even for values of $\rho_2 > 1$, especially for σ_f and Q_t and for a dielectric inner sphere. The maximum value of ρ_2 used in our figures for the parameters of Table I and $R_2/R_1 = 0.02$ is 0.140, keeping the percent error low enough in each case. The technique of the present paper is also applicable to complex scatterers other than sphere.

Appendix A

The expansion coefficients $S_{mn}(O)$, $T_{mn}(O)$, $F_{mn}(O)$, $G_{mn}(O)$ ((3)–(6)) for the problem of the scattering of a plane electromagnetic wave from an homogeneous dielectric sphere of radius R_1, with parameters ϵ_2, μ_2, k_2, surrounded by an homogeneous medium of infinite extent, with parameters ϵ_1, μ_1, k_1, are calculated easily by the satisfaction of the boundary conditions at $r_1 = R_1$ and have the following expressions ($m = -1, 1, n \geqq 1$)

$$S_{mn}(O) = \frac{i\mu_2(E_{ox} - imE_{oy})g_{mn}}{\rho_1[\mu_1 h_n(\rho_1)j_n^d(\rho_3) - \mu_2 h_n^d(\rho_1)j_n(\rho_3)]} \quad \text{(A1)}$$

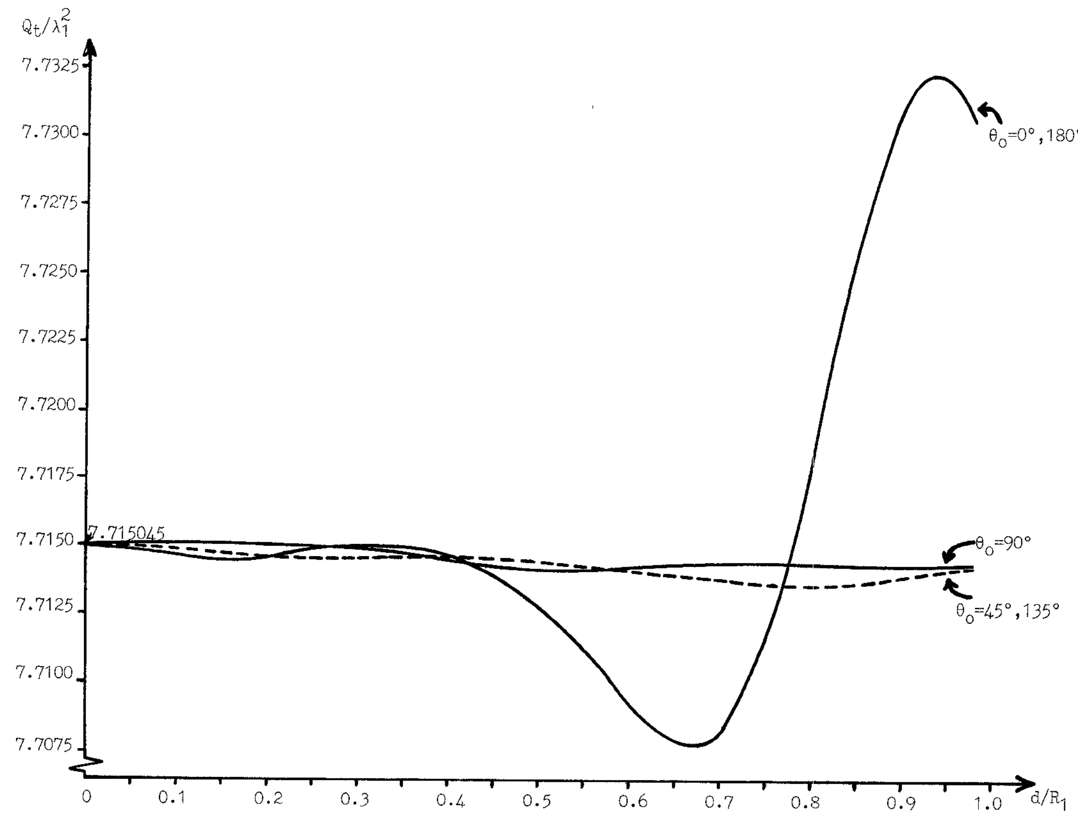

Fig. 12. Total scattering cross-section for $\epsilon_2/\epsilon_1 = 5.5$, $\epsilon_3/\epsilon_1 = 2.54$, $R_1/\lambda_1 = 0.7$, $R_2/R_1 = 0.02$, $\phi_o = 45°$, $E_{ox} = E_{oy} = 1/\sqrt{2}$ (dielectric inner sphere).

TABLE I
PERCENT ERROR BETWEEN OUR RESULTS AND THE EXACT ONES FOR $d = 0$

		Metallic inner sphere ($\varepsilon_2/\varepsilon_1$=2.54, R_1/λ_1=0.7)			Dielectric inner sphere ($\varepsilon_2/\varepsilon_1$=2.54, $\varepsilon_3/\varepsilon_1$=5.5, R_1/λ_1=0.7)		
ρ_2	R_2/R_1	σ_b	σ_f	Q_t	σ_b	σ_f	Q_t
0.210	0.03	-2.8×10^{-3}	2.54×10^{-3}	1.17×10^{-3}	-7.6×10^{-4}	2.8×10^{-4}	1.2×10^{-4}
0.350	0.05	-8.37×10^{-3}	3.52×10^{-2}	1.55×10^{-2}	-7.26×10^{-3}	3.76×10^{-3}	1.67×10^{-3}
0.491	0.07	0.147	0.196	0.082	-0.0272	0.021	9.31×10^{-3}
0.631	0.09	1.37	0.677	0.256	-0.0525	0.0767	0.0329
0.771	0.11	6.37	1.54	0.462	-0.023	0.212	0.0881
1.051	0.15	36.7	0.186	-1.46	0.926	0.928	0.347
1.332	0.19	-7.6	-9.5	-7.78	4.2	2.2	0.65
1.612	0.23	-167.9	-22.6	-16.1	3.2	2.8	0.265
1.893	0.27	-231.9	-39.1	-26.6	-15	0.728	-1.5
2.173	0.31	-241.9	-60.5	-40.0	-37.5	-4.1	-4.3

$$T_{mn}(O) = \frac{imk_1k_2\mu_2(E_{ox} - imE_{oy})g_{mn}}{\rho_1[k_1^2\mu_2 h_n(\rho_1)j_n^d(\rho_3) - k_2^2\mu_1 h_n^d(\rho_1)j_n(\rho_3)]} \quad \text{(A2)}$$

$$F_{mn}(O) = (E_{ox} - imE_{oy})g_{mn} \cdot \frac{\mu_1 j_n(\rho_1)j_n^d(\rho_3) - \mu_2 j_n^{d}(\rho_1)j_n(\rho_3)}{\mu_1 h_n(\rho_1)j_n^d(\rho_3) - \mu_2 h_n^d(\rho_1)j_n(\rho_3)} \quad \text{(A3)}$$

$$G_{mn}(O) = m(E_{ox} - imE_{oy})g_{mn} \cdot \frac{\epsilon_1 j_n(\rho_1)j_n^d(\rho_3) - \epsilon_2 j_n^d(\rho_1)j_n(\rho_3)}{\epsilon_1 h_n(\rho_1)j_n^d(\rho_3) - \epsilon_2 h_n^d(\rho_1)j_n(\rho_3)} \quad \text{(A4)}$$

where ρ_1, ρ_3 and g_{mn} are given in (25). In (A1) and (A2) the Wronskian relation $j_n(\rho_1)h_n^d(\rho_1) - j_n^d(\rho_1)h_n(\rho_1) = i/\rho_1$ is used.

APPENDIX B

The symbols $a(m, n|-\mu, s|p)$, $a(n, s, p)$, etc., appearing in (13), (14), are calculated along steps described in the Appendix of [15], which will not be repeated here. There, are also found certain special values of them, as well as of $A_{\mu s}^{mn}$ and $B_{\mu s}^{mn}$, necessary for the solution of the present problems. Some other special values of these symbols useful in the present work and not appearing in [15], are related to the ones found there by the following very simple equations:

$$a(\pm 1, 1|m, n|n-1) = a(m, n|\pm 1, 1|n-1),$$

$$a(\pm 1, 1|m, n|n+1) = a(m, n|\pm 1, 1|n+1) \tag{B1}$$

$$a(0, 1|m, n|n \pm 1) = a(m, n|0, 1|n \pm 1) \tag{B2}$$

$$b(1, n, n+2) = b(n, 1, n+2) = 0 \tag{B3}$$

$$a(\pm 1, 1|m, n|n, n-1)b(1, n, n) = (-1)^{n+1}\tau_n a(m, n|\pm 1, 1|n, n-1)b(n, 1, n) \tag{B4}$$

$$a(0, 1|m, n|n, n-1)b(1, n, n) = (-1)^{n+1}\tau_n a(m, n|0, 1|n, n-1)b(n, 1, n) \tag{B5}$$

$$C_{mn}^{u1} = (-1)^u \tau_n A_{-u1}^{-mn}, \qquad D_{mn}^{u1} = (-1)^u \tau_n B_{-u1}^{-mn}, \quad (m = -1, 1, u = -1, 0, 1, n \geqq 1) \tag{B6}$$

where

$$\tau_n = -\frac{2(2n+1)}{3n(n+1)}. \tag{B7}$$

References

[1] A. A. Kishk, R. P. Parrikar and A. Z. Elsherbeni, "Electromagnetic scattering from an eccentric multilayered circular cylinder," *IEEE Trans. Antenn. Propagat.*, vol. 40, pp. 295–303, Mar. 1992.

[2] A. A. Kishk, A. W. Glisson and P. M. Goggans, "Scattering from conductors coated with materials of arbitrary thickness," *IEEE Trans. Antenn. Propagat.*, vol. 40, pp. 108–112, Jan. 1992.

[3] E. N. Vasilev, V. V. Solodukhov, and A. I. Fedorenko, "The integral equation method in the problem of electromagnetic waves diffraction by complex bodies," *Electromagnetics*, vol. 11, pp. 161–182, 1991.

[4] H. A. Ragheb, L. Shafai, and M. Hamid, "Plane wave scattering by a conducting elliptic cylinder coated by a nonconfocal dielectric," *IEEE Trans. Antenn. Propagat.*, vol. 39, pp. 218–223, Feb. 1991.

[5] J. L. Volakis and H. H. Syed, "Application of higher order boundary conditions to scattering by multilayer coated cylinders," *J. Electromagnetic Waves and Applications*, vol. 4, no. 12, pp. 1157–1180, 1990.

[6] J. Baldauf, S. W. Lee, H. Ling, and R. Chou, "On physical optics for calculating scattering from coated bodies," *J. Electromagnetic Waves and Applications*, vol. 3, no. 8, pp. 725–746, 1989.

[7] J. A. Roumeliotis, J. G. Fikioris, and G. P. Gounaris, "Electromagnetic scattering from an eccentrically coated infinite metallic cylinder," *J. Appl. Phys.*, vol. 51, pp. 4488–4493, Aug. 1980.

[8] J. A. Roumeliotis and J. G. Fikoris, "Scattering of plane waves from an eccentrically coated metallic sphere," *J. Franklin Inst.*, vol. 312, pp. 41–59, July 1981.

[9] N. B. Kakogiannos and J. A. Roumeliotis, "Electromagnetic scattering from an infinite elliptic metallic cylinder coated by a circular dielectric one," *IEEE Trans. Microwave Theory Tech.*, vol. 38, pp. 1660–1666, Nov. 1990.

[10] J. A. Roumeliotis and N. B. Kakogiannos, "Scattering from an infinite cylinder of small radius embedded into a dielectric one," *IEEE Trans. Microwave Theory Tech.*, vol. 42, pp. 463–470, 1994.

[11] J. H. Bruning and Y. T. Lo, "Multiple scattering of EM waves by spheres. Part I-multipole expansion and ray-optical solutions," *IEEE Trans. Antennas Propagat.*, vol. 19, pp. 378–390, May 1971.

[12] ——, "Multiple scattering of EM waves by spheres. Part II—numerical and experimental results," *IEEE Trans. Antennas, Propagat.*, vol. 19, pp. 391–400, May 1971.

[13] J. A. Stratton, *Electromagnetic Theory*. New York: McGraw-Hill, 1941.

[14] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*. New York: McGraw-Hill, 1953.

[15] J. A. Roumeliotis, J. D. Kanellopoulos, and J. G. Fikioris, "Resonant frequencies in an electromagnetic spherical cavity with an eccentric inner electrically small sphere," *Electromagnetics*, vol. 12, no. 2, pp. 155–170, 1992.

[16] O. R. Cruzan, "Translational addition theorems for spherical vector wave functions," *Q. Appl. Math.*, vol. 20, pp. 33–40, 1962.

[17] W. C. Chew, *Waves and Fields in Inhomogeneous Media*. New York: Van Nostrand Reinhold, 1990.

[18] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. New York: Dover Publ., 1972.

[19] J. Van Bladel, *Electromagnetic Fields*. New York: McGraw-Hill, 1964.

[20] R. F. Harrington, *Time Harmonic Electromagnetic Fields*. New York: McGraw-Hill, 1961.

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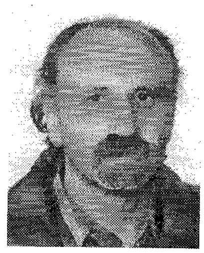

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